The Inertial Forces / Test Particle Motion Game

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The somewhat fragmented body of current literature analyzing the properties of test particle motion in static and stationary spacetimes and in general spacetimes is pulled together and clarified using the framework of gravitoelectromagnetism.

1 Introduction

During the past decade a long list of authors have studied various properties of test particle motion in static or stationary axisymmetric spacetimes, usually in "uniform circular motion" along an integral curve of a timelike Killing vector field associated with this symmetry. Many of these authors have decomposed the acceleration vector field along the test world line with respect to a family of observers defined on most of the spacetime which are also in uniform circular motion. This acceleration vector is orthogonal to the cylindrical world sheet symmetry group orbits containing the circular world lines, the case of purely transverse relative acceleration with respect to any of these observer families, and so the observer spatial projection acts as the identity on the acceleration vector. Thus in this special case, the relative observer / relative motion orthogonal splitting $4 \to 3+1 \to (2+1)+1$ of the full tangent space first into the local rest space and local time direction of the observer and then of the local rest space into its transverse and longitudinal subspaces relative to the direction of relative motion of the test particle provides a representation of the full acceleration vector itself rather than just of its observer spatial projection. However, the relative observer approach has the advantage over the direct spacetime approach of offering a directly interpretable physical relative velocity variable to parametrize the circular motion, whereas the spacetime approach is limited to the use of the coordinate angular velocity variable whose interpretation (as a speed to

be compared to the speed of light) depends on further quantities.

Some authors have attempted to generalize a conformal variation of this relative observer / relative motion splitting of the acceleration of the acceleration vector of a test particle from these special symmetry conditions to general motion in general spacetimes. However, the general language of gravitoelectromagnetism, which provides a systematic approach to work much of which is many decades old, already provides unambiguously all possible such acceleration decompositions in a general spacetime. For a rotating black hole spacetime where there is a natural slicing by the integral hypersurfaces of the local rest spaces of the locally nonrotating (zero angular momentum) observers and a natural threading by the world lines of the static Killing observers which are at rest with respect to spatial infinity, there are three different descriptions (hypersurface, threading, and slicing) and 3 natural observer time derivatives, one of which does not commute with index shifting leading to 4 distinct time derivatives of the contravariant and covariant forms of a vector field, and therefore to 12 different acceleration decompositions, which double in number when one includes the Fermat's principle inspired optical conformal transformation variation of this decomposition. The utility of any of these 24 possibilities for describing test particle motion in any spacetime remains to be justified in particular applications.

In the very special case of a static axisymmetric spacetime, the various points of view and observer time derivatives compatible with the symmetry each collapse from three to one in number, and these 24 descriptions reduce to 2 in number, the standard relative observer approach and its optical conformal variation. Each of these seem useful in understanding certain aspects of the qualitative behavior of test particle motion, and the latter is clearly an elegant geometrization of the relative motion of massive and massless test particles.

The present article summarizes this situation in an attempt to bring more understanding among the current groups in this field of each other's distinct points of view.

2 The Roster of Players

A number of different approaches have been taken in studying the properties of circular orbits in static and stationary spacetimes. One approach followed by De Felice, Semerák, Page, and others attacks the problem from the spacetime point of view, while others use a space-plus-time decomposition of the 4-acceleration with respect to a family of test observers, leading to an interpretation involving inertial forces due to their motion. Of the latter Abramowicz et al have been evolving a generalization of such an approach from the special static axisymmetric case up to general spacetimes, while Bini, Carini, and Jantzen have specialized gravitoelectromagnetism, the relativity of observer splitting formalisms, from the case of general spacetimes down to stationary and axisymmetric ones. The tools of this last approach provide a necessary clarification and comparison of all the work in this area. In particular the gradual reorientation of the Abramowicz et al 4-acceleration decomposition from the static case to general spacetimes is shown to have as its target the conformal modification inspired by Fermat's principle of the

standard space-plus-time splittings of the various observer congruence approaches (congruence, threading, hypersurface, slicing) in use today.

The players in the current inertial forces / test particle motion game fall into a number of groups.

Abramowicz et al [1–30]

Abramowicz and Lasota examined the roots of their later work in 1974 but did not really begin studying inertial, especially centrifugal, forces in static and stationary spacetimes until Abramowicz and Lasota [2] (1986) and Abramowicz, Carter, and Lasota [3] in 1988 (as an alternative to an earlier approach of De Felice, for example, as they explicitly note), and were then joined in continuing this work and extending it to general spacetimes in a series of applications and sequential corrections by Prasanna, Chakrabarti, Bičák, Miller, Stuchlík, Nurowski, Wex, Sonego, Lanza, Massar, and Iyer.

Bini, Carini, and Jantzen [31–47]

Jantzen and Carini (1991) were later joined by Bini in studying the larger picture of the relativity of spacetime splitting formalisms and inertial forces in test particle motion, which they called gravitoelectromagnetism. More recently they have specialized this work to some familiar stationary axisymmetric spacetimes.

De Felice and Usseglio-Tomasset [48–60]

De Felice (1971) began by studying properties of test particle motion in black hole spacetimes and did some preliminary work looking at the inertial force analogy in 1975, but did not use the inertial force language, instead relying on the angular velocity variable to express spacetime results for circular orbits. With Usseglio-Tomasset in 1991 he began overlapping with the Abramowicz group work, finally joining with Semerák.

Vishveshwara and Iyer [61–67]

Vishveshwara started with collaborators Honig and Schücking (1974) and Greene and Schücking (1975), examining the stationary rest frame in black hole spacetimes and was later joined by Iyer (1993) in examining test particle motion in Killing submanifolds using the spacetime Frenet-Serret formalism and then with Nayak made contact with the Abramowicz inertial force ideas.

Semerák [68–82]

Semerák started by examining stationary observer families in black hole spacetimes in 1993, clearly making contact with the work of de Felice, and then continued on to inertial forces and test particle motion in general and in black hole spacetimes starting in 1995, in the context of the Abramowicz et al and Bini-Carini-Jantzen work. (See also earlier work by Tsoubelis, Economou, and Stoghianidis [80,81] on spinning test particles in black hole spacetimes.) Page continued studying properties of stationary observer congruences in the same spirit. [82]

Barrabès, Boisseau, and Israel [83]
Along the way Barrabès, Boisseau, and Israel (1995) injected some useful perspectives on the issues of inertial forces in black hole spacetimes from the hypersurface point of view.

Rindler and Perlick [84]
Rindler and Perlick (1990) introduced a nice analysis of circular orbits in stationary axisymmetric spacetimes which proved useful in some of the above work. This was followed by interesting work by Perlick on Fermat's principle in general relativity (see, e.g., Møller [89]) and related ideas. (See also Rindler's recent article. [90])

Most of this work using a relative observer analysis has its roots in the literature dating back to the fifties, although there seems to be somewhat of a literature horizon effect in the bibliographies of the current generation of articles. An extensive bibliography is given elsewhere. [32] Similar questions about properties of test particle motion are also studied directly from the spacetime point of view without direct reference to a test observer family as already noted above. This article will discuss the situation from the observer point of view. The spacetime metric g is assumed to have signature `(-+++)` and $||X|| = |g(X,X)|^{1/2}$ will denote the length of a vector field, while $X^\flat$ and $X^\sharp$ will denote the totally covariant and totally contravariant forms of a tensor field obtained by index shifting with the spacetime metric.

3 The Game: Observing Test Particle Motion and Spin Precession Effects

One starts with 1) a congruence of test observer world lines with 4-velocity field u covering the region of interest in a given spacetime and then analyzes the properties of 2) individual test particle (timelike or null) world lines with 4-velocity U (timelike case) or 4-momentum P (null case) defined only along each such world line.

For a single such test world line, attention is thus confined to a "**relative observer world 2-sheet**" in spacetime illustrated in Figure 1 representing the sheaf of observer world lines which cross the test particle world line. This world sheet projects down to a curve in the quotient space of observer world lines representing the trajectory in the test observer "**space**." At each point of the test particle world line, the tangent plane to this 2-sheet is the "**relative observer 2-plane**" spanned by u and U or u and P also containing the unit vectors representing the directions of relative motion. In the timelike case, for example, as illustrated in Figure 2, one has the reciprocal pair of orthogonal decompositions

$$\begin{aligned} U &= \gamma(U,u)[u+\nu(U,u)]\ , \qquad \hat\nu(U,u) = ||\nu(U,u)||^{-1}\nu(U,u)\ , \\ u &= \gamma(u,U)[U+\nu(u,U)]\ , \qquad \hat\nu(u,U) = ||\nu(u,U)||^{-1}\nu(u,U)\ , \end{aligned} \tag{1}$$

while in the null case one has only

$$P = E(P,u)[u+\hat\nu(P,u)]\ , \tag{2}$$

where the hat notation indicates unit vectors. One can also introduce the momentum representation of this orthogonal decomposition for the timelike case (for convenience set the rest mass $m = 1$ so that $P = mU = U$)

$$P = E(U,u)u + p(U,u) = \gamma(U,u)u + ||p(U,u)||\hat{\nu}(U,u) \ . \tag{3}$$

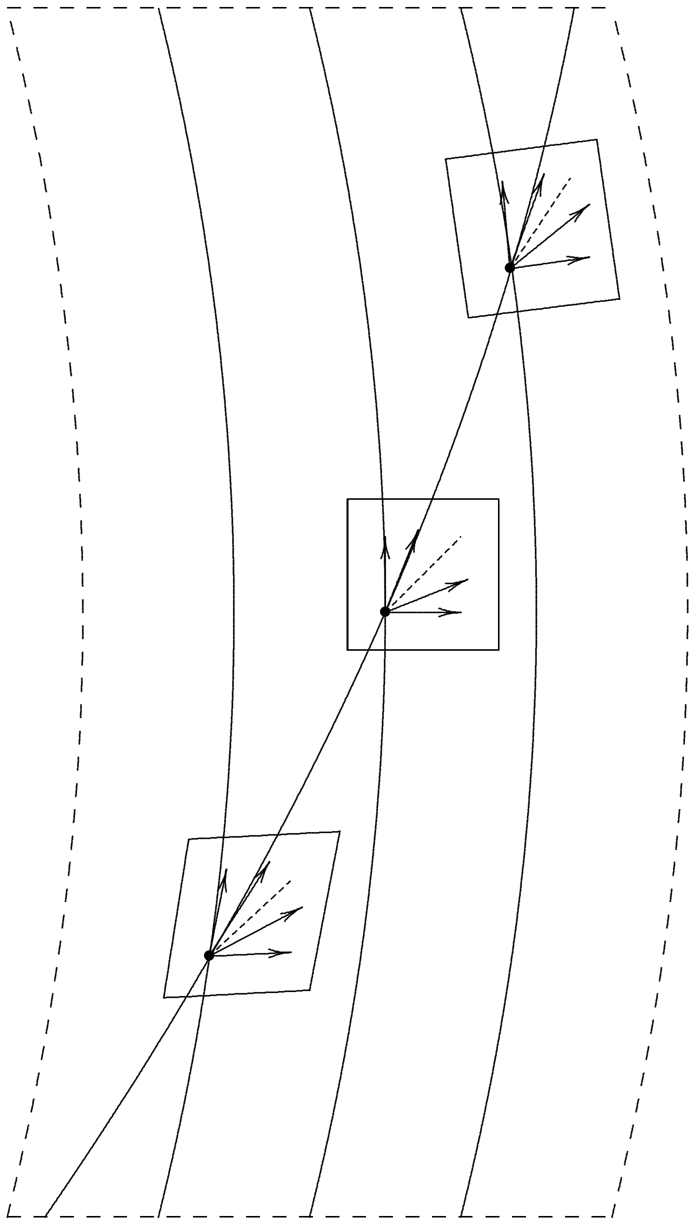

Figure 1: The relative observer world sheet of u and U.

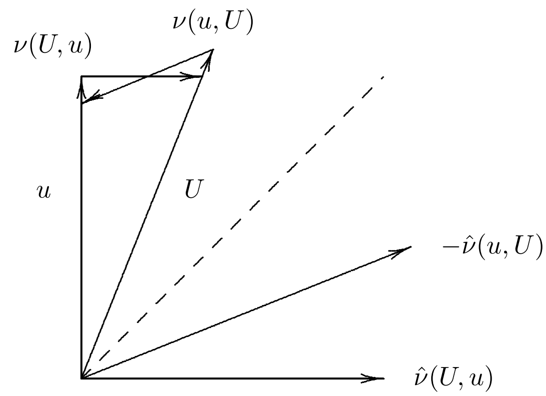

Figure 2: The relative observer plane of u and U. The unit vector $\hat\nu(U,u)$ gives the direction of the longitudinal subspace of the local rest space LRS_u relative to U, while $\hat\nu(u,U)$ does the same for LRS_U.

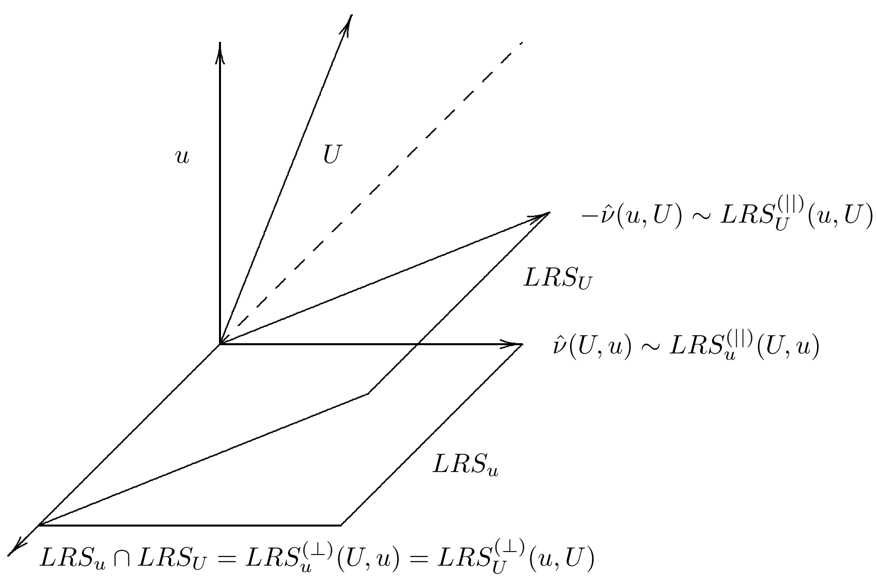

Figure 3: The relative motion orthogonal decomposition of the local rest spaces of u and U. The intersection of the two local rest spaces is the common 2-dimensional transverse subspace of the tangent space.

Table 1: Summary of the key concepts underlying the natural mathematical structure of this problem.

observer congruence:	orthogonal projection, kinematical quantities
$+$	
test particle world line:	intrinsic derivative
$\Longrightarrow$	
relative observer world sheet:	projected intrinsic derivatives

In the timelike case one has a further "**relative motion orthogonal decomposition**" illustrated in Figure 3 of the two local rest spaces LRS_u and LRS_U into the common 2-dimensional subspace $LRS_u \cap LRS_U$ transverse to the direction of relative motion and its 1-dimensional (longitudinal) orthogonal complement along the direction of relative motion in each such rest space. In the null case one has only the orthogonal decomposition of LRS_u into the 1-dimensional longitudinal subspace along the direction of relative motion and its 2-dimensional transverse orthogonal complement.

Spin precession effects for a test particle are studied via a vector S belonging to LRS_U undergoing Fermi-Walker transport along the test world line in the timelike case. The behavior of the world line itself as seen by the observer family requires the choice of a geometric temporal derivative operator along the test world line, equivalent to the choice of a spatial frame along that world line (having zero derivative), in order to measure changes in the components of the measured fields as they evolve along the world line, given a choice of parametrization for it. A number of such geometric derivatives along a curve are available, leading to collection of quantities that are defined by derivative formulas, depending on the choice of this derivative.

This mathematical setting has a natural mathematical structure which requires no invention, which is unambiguous, and which has a direct physical interpretation. The key concepts are summarized in Table 1.

4 Inertial forces: Measuring the 4-Acceleration/4-Force Equation

The 4-acceleration $a(U) = DU/d\tau_U$ of a timelike test particle parametrized by the proper time τ_U is the intrinsic derivative of the 4-velocity along its world line. The equation of motion for the test particle is the 4-acceleration/4-force equation

$$a(U) = f(U) \ . \tag{4}$$

This is measured by the observer family by orthogonally projecting it into a family of spatial fields, in this case a scalar and a spatial vector. The temporal projection along u leads to the evolution equation for the observed energy of the test particle along its world line, while the spatial projection orthogonal to u leads to the evolution equation for the observed 3-momentum of the particle along its world line, with the kinematical quantities describing the motion of the family of observers entering these measured equations as "**inertial forces**." The spatial projection operator

$P(u)$ acts on a tensor by contraction of all indices with the corresponding projection tensor, here identified with the same symbol $P(u) = Id + u \otimes u^\flat$. The covariant form of this mixed index tensor is the spatial metric $h(u) = P(u)^\flat = P(u)g$.

Dividing the measured spatial projection equation by the gamma factor of the test particle in the timelike case, one may absorb this factor into a reparametrization of the world line by the sequence of observer proper times $\tau_{(U,u)}$ along it rather than by the test particle's own proper time τ_U, related by $d\tau_{(U,u)}/d\tau_U = \gamma(U,u)$. The result

$$\gamma(U,u)^{-1}P(u)a(U) = \gamma(U,u)^{-1}P(u)f(U) \tag{5}$$

is directly equivalent to

$$D_{\rm (fw)}(U,u)\, p(U,u)/d\tau_{(U,u)} - F^{\rm (G)}_{\rm (fw)}(U,u) = F(U,u) \ , \tag{6}$$

where $P(u)D/d\tau_U = D_{\rm (fw)}(U,u)/d\tau_U$ is the spatial Fermi-Walker intrinsic derivative along the test world line and $F(U,u) = \gamma(U,u)^{-1}P(u)f(U)$ is the measured 3-force. The term appearing with a minus sign on the left hand side of this projected, rescaled 4-acceleration/4-force equation

$$\begin{aligned} F^{\rm (G)}_{\rm (fw)}(U,u) &= -\gamma(U,u)^{-1}P(u)D\, u/d\tau_U = -D_{\rm (fw)}(U,u)\, u/d\tau_{\rm (U,u)} \\ &= \gamma(U,u)[-a(u) + \{-\omega(u)+\theta(u)\} \llcorner \nu(U,u)] \\ &= \gamma(U,u)[g(u) + H_{\rm (fw)}(u) \llcorner \nu(U,u)] \ , \end{aligned} \tag{7}$$

arising from the intrinsic derivative of u along U with respect to the test particle proper time τ_U, may be moved to the right hand side with a positive sign in order to be interpreted as inertial forces due to the motion of the observers themselves. These inertial forces involve the kinematical quantities of the observer congruence, namely the acceleration vector field $a(u)$ (leading to a gravitoelectric field and a spatial gravitoelectric gravitational force) and their vorticity/rotation $\omega(u)$ and expansion $\theta(u)$ mixed tensor fields (leading to a gravitomagnetic vector field and tensor field and a Coriolis or gravitomagnetic force linear in the relative velocity, modulo the gamma factor). This reflects the observer measurement orthogonal decomposition at the derivative level.

The further relative motion orthogonal decomposition of the observer local rest spaces LRS_u along the test particle world line may be similarly used to decompose the temporal derivative of the 3-momentum into a longitudinal relative acceleration term and a transverse one

$$\begin{aligned} p(U,u) &= ||p(U,u)||\hat\nu(U,u) \ , \\ D_{\rm (fw)}(U,u)\, p(U,u)/d\tau_{(U,u)} &= [D_{\rm (fw)}(U,u)||p(U,u)||/d\tau_{(U,u)}]\hat\nu(U,u) \\ &\quad +||p(U,u)||\, D_{\rm (fw)}(U,u)\, \hat\nu(U,u)/d\tau_{(U,u)} \ , \end{aligned} \tag{8}$$

the latter of which may be re-expressed in the form

$$\begin{aligned} [D_{\rm (fw)}(U,u)\, p(U,u)/d\tau_{(U,u)}]^{(\perp)} &= \gamma(U,u)a^{(\perp)}_{\rm (fw)}(U,u) \ , \\ a^{(\perp)}_{\rm (fw)}(U,u) &= \nu(U,u)^2[D_{\rm (fw)}(U,u)\, \hat\nu(U,u)/d\ell_{(U,u)}] \ , \end{aligned} \tag{9}$$

where the derivative has been reparametrized using the spatial arclength parametrization, corresponding to the relationship $d\ell_{(U,u)}/d\tau_{(U,u)} = ||\nu(U,u)||$, and where

$$a_{(\mathrm{fw})}(U,u) = D_{(\mathrm{fw})}(U,u)\,\nu(U,u)/d\tau_{(U,u)}$$

is the Fermi-Walker relative acceleration. Apart from the additional gamma factor, the second term in the decomposition of the spatial momentum rate of change above is the transverse part of the relative acceleration, namely the Fermi-Walker relative centripetal acceleration. The factor in square brackets in the latter expression may be expressed in terms of its direction (relative normal) and magnitude (relative curvature)

$$D_{(\mathrm{fw})}(U,u)\,\hat{\nu}(U,u)/d\ell_{(U,u)} \quad = \quad \kappa_{(\mathrm{fw})}(U,u)\,\hat{\eta}_{(\mathrm{fw})}(U,u)\ , \tag{10}$$

with the reciprocal of the magnitude defining the Fermi-Walker relative curvature $\rho_{(\mathrm{fw})}(U,u)$ of the world line, in terms of which the Fermi-Walker relative centripetal acceleration takes its usual form familiar from nonrelativistic mechanics

$$a^{(\perp)}_{(\mathrm{fw})}(U,u) = [\nu(U,u)^2/\rho_{(\mathrm{fw})}(U,u)]\hat{\eta}_{(\mathrm{fw})}(U,u)\ . \tag{11}$$

Continuing this leads to the Fermi-Walker relative bi-normal $\hat{\xi}_{(\mathrm{fw})}(U,u)$ and torsion $\tau_{(\mathrm{fw})}(U,u)$ for the test world line and the remaining relative Frenet-Serret equations in complete analogy with the geometry of a curve in a three-dimensional Riemannian manifold. [45] These remaining relative Frenet-Serret equations are given by

$$\begin{aligned}
\frac{D_{(\mathrm{fw})}(U,u)}{d\ell_{(U,u)}}\hat{\eta}_{(\mathrm{fw})}(U,u) \quad &= \quad -k_{(\mathrm{fw})}(U,u)\hat{\nu}(U,u) + \tau_{(\mathrm{fw})}(U,u)\hat{\xi}_{(\mathrm{fw})}(U,u)\ ,\\
\frac{D_{(\mathrm{fw})}(U,u)}{d\ell_{(U,u)}}\hat{\xi}_{(\mathrm{fw})}(U,u) \quad &= \quad -\tau_{(\mathrm{fw})}(U,u)\hat{\eta}_{(\mathrm{fw})}(U,u)\ ,
\end{aligned} \tag{12}$$

or

$$\frac{D_{(\mathrm{fw})}(U,u)}{d\ell_{(U,u)}}E_{(\mathrm{fw})}(U,u)_a = \omega_{(\mathrm{fw})}(U,u)\times_u E_{(\mathrm{fw})}(U,u)_a\ , \tag{13}$$

where

$$\{E_{(\mathrm{fw})}(U,u)_a\} = \{\hat{\nu}(U,u), \hat{\eta}_{(\mathrm{fw})}(U,u), \xi_{(\mathrm{fw})}(U,u) = \hat{\nu}(U,u)\times_u \hat{\eta}_{(\mathrm{fw})}(U,u)\} \tag{14}$$

and $\omega_{(\mathrm{fw})}(U,u) = \tau_{(\mathrm{fw})}(U,u)\hat{\nu}(U,u) + k_{(\mathrm{fw})}(U,u)\hat{\xi}_{(\mathrm{fw})}(U,u)$.

The centripetal acceleration term may also be moved to the right hand side of this equation with a minus sign to be interpreted as a "**centrifugal force**," which might be useful for the case of relative motion at fixed speed where the longitudinal acceleration vanishes (purely transverse relative motion), in order to interpret the acceleration/force equation as a balance of spatial forces. Although this transverse term also belongs to the test particle local rest space LRS_U, it does not directly correspond to a force in that frame, since it only has meaning in the context of the entire observer local rest space, where the remaining terms in this projected acceleration/force equation live in general, and is also scaled by a factor of gamma compared to a component of the 4-force in the test particle local rest space.

The longitudinal 3-acceleration term may also be moved to the right hand side with a minus sign and interpreted as an "**Euler**" force in the terminology of Abramowicz et al, so that the entire equation may be thought of as a balance of 3-forces from the point of view of the family of observers along the test world line.

5 Variations of the Projected Intrinsic Derivative

Observer measurements of tensor fields and tensor field differential operators involve the observer measured derivative operators

$$\begin{aligned} \nabla(u)X &= P(u)\nabla X \ , \\ \nabla_{(\mathrm{fw})}(u)X &= P(u)\nabla_u X \ , \quad \nabla_{(\mathrm{lie})}(u)X = P(u)\pounds_u X \equiv \pounds(u)_u X \ . \end{aligned} \quad (15)$$

The first is the spatial covariant derivative, for which the spatial metric $h(u)$ is covariant constant. In addition to the last two of these which are temporal derivatives, namely the spatial Fermi-Walker derivative and the spatial Lie derivative along u, one may introduce a third temporal derivative, the corotating spatial Fermi-Walker derivative, so that the three are related as follows when acting on a vector field X

$$\begin{aligned} \nabla_{(\mathrm{cfw})}(u)X &= \nabla_{(\mathrm{fw})}(u)X + \omega(u) \llcorner X \\ &= \nabla_{(\mathrm{lie})}(u)X + \theta(u) \llcorner X \ , \end{aligned} \quad (16)$$

differing among themselves only by the action of a linear transformation of the observer local rest space. The spatial metric has vanishing derivative with respect to only the Fermi-Walker and corotating Fermi-Walker such derivatives, while it has a nonvanishing derivative $\nabla_{(\mathrm{lie})}(u)\, h(u) = 2\theta(u)^\flat$ so that index-shifting with the spatial metric does not commute with this latter derivative unless the expansion tensor vanishes, as occurs in the case of a stationary Killing observer congruence in a stationary spacetime, where the Lie and corotating spatial temporal derivatives along the observer congruence coincide.

If X is a tensor field on spacetime, its intrinsic derivative along a timelike test particle world line with 4-velocity U equals its covariant derivative along U

$$DX/d\tau_U = \nabla_U X = \gamma(U,u)[\nabla_u + \nabla_{\nu(U,u)}]X \ . \quad (17)$$

Spatially projecting this leads to the Fermi-Walker spatial intrinsic derivative along the world line. Changing the parametrization to observer proper time as well leads to

$$D_{(\mathrm{fw})}(U,u)\, X/d\tau_{(U,u)} = \nabla_{(\mathrm{fw})}(u)X + \nabla(u)_{\nu(U,u)}X \ . \quad (18)$$

This may be extended to define two other spatial intrinsic derivatives along the test world line by replacing the Fermi-Walker temporal derivative by the corotating one or the spatial Lie derivative along u

$$D_{(\mathrm{tem})}(U,u)\, X/d\tau_{(U,u)} = \nabla_{(\mathrm{tem})}(u)X + \nabla(u)_{\nu(U,u)}X \ , \quad \text{tem=fw,cfw,lie} \ , \quad (19)$$

which are then related to each other in the same way as the observer temporal derivatives that they generalize. For example, for a vector field X one has

$$\begin{aligned} D_{(\mathrm{cfw})}(U,u)\, X/d\tau_{(U,u)} &= D_{(\mathrm{fw})}(U,u)\, X/d\tau_{(U,u)} + \omega(u) \llcorner X \\ &= D_{(\mathrm{lie})}(U,u)\, X/d\tau_{(U,u)} + \theta(u) \llcorner X \ , \end{aligned} \quad (20)$$

In order to express these operators in a form in which they may be directly applied to a field defined only along the world line, one obtains a formula analogous to the one for the intrinsic derivative itself with respect to some choice of spacetime frame, involving the components of the connection with respect to that frame

$$DX^\alpha/d\tau_U = dX^\alpha/d\tau_U + \Gamma^\alpha{}_{\beta\gamma}U^\beta X^\gamma \ . \tag{21}$$

For the projected intrinsic derivatives, reparametrized to observer proper time, the appropriate projections of these connection component terms are joined by additional kinematical terms in the "cfw" and "lie" cases

$$D_{(\mathrm{tem})}\, X^\alpha/d\tau_{(U,u)} = dX^\alpha/d\tau_{(U,u)} + \Delta_{(\mathrm{tem})}(u)^\alpha{}_{\beta\gamma}U^\beta X^\gamma/\gamma(U,u)\ ,\ {}_{\mathrm{tem=fw,cfw,lie}}\ . \tag{22}$$

The projections of the connection component terms themselves involve the spatial connection components (associated with the spatial metric $h(u)$) and kinematical terms. However, only the Fermi-Walker and corotating Fermi-Walker such derivatives commute with index-shifting, while the Lie such derivative of the spatial metric is

$$D_{(\mathrm{lie})}(U,u)\, h(u)/d\tau_{(U,u)} = 2\theta(u)^\flat \ , \tag{23}$$

so an additional expansion term appears when shifting indices on an object being differentiated by this derivative, so that derivatives of the contravariant and covariant forms of a vector field differ by an expansion term. Another consequence of this is that its derivative of a unit spatial vector field is not orthogonal to the original vector field in general

$$\hat\nu(U,u)\cdot D_{(\mathrm{lie})}(U,u)\,\hat\nu(U,u)/d\tau_{(U,u)} = -||\nu(U,u)||^{-1}\theta(u)^\flat(\hat\nu(U,u),\hat\nu(U,u))\ , \tag{24}$$

in contrast with the situation for the other two derivatives which respect spatial inner products.

Thus for a given observer congruence, one has four different natural "**relative Frenet-Serret structures**" along a test particle world line corresponding to the three temporal derivative operators with the index-shifting complication, and four different decompositions of the acceleration/force equation and its respective terms, corresponding to different pairs of numerical coefficients of the vorticity and expansion terms in the gravitomagnetic field tensor.

For a stationary spacetime with a stationary observer congruence, it is most natural to use the Lie spatial intrinsic derivative, since it is the one most closely connected to the spatial geometry without additional kinematical linear transformations of the spatial tangent space. If the observer congruence is along a Killing direction, then this derivative also commutes with index-shifting by the spatial metric, and the Lie relative Frenet-Serret structure on the observer quotient manifold coincides with the Riemannian Frenet-Serret structure of the time-independent spatial metric there. If the relative observer world sheet is a group orbit in the stationary axisymmetric case (circular orbits), then the Lie spatial intrinsic derivative of the unit spatial velocity is also transverse.

The most interesting nontrivial case to study along these lines is that of the (noncircular) spherical accelerated orbits and geodesics in a Kerr black hole spacetime. These do not follow Killing trajectories and in general are not characterized by transverse relative acceleration.

6 Further Complications

Usually spacetime splittings are facilitated by the use of adapted coordinate systems, or more generally by equivalence classes of such coordinate systems which define a parametrized nonlinear reference frame. This "**full splitting of spacetime**," as opposed to the partial splitting provided by an observer congruence alone, involves the independent structures of a slicing of the spacetime by a family of hypersurfaces parametrized by a time function and a threading of this family by a transversal congruence of curves. Depending on the causality properties of these two components one can adopt one of two possible points of view using this structure.

If the congruence is timelike, one can interpret it as the observer congruence, leading to the threading point of view. If the slicing is spacelike, one can interpret its field of unit normals as the observer congruence, leading to the hypersurface point of view. One can also adopt the slicing point of view using the threading to evolve the fields in spacetime rather than the observer congruence itself. All the geometry of the threading and hypersurface points of view is identical with that already discussed above for the observer congruence, while some new features arise from the bi-conguence approach of the slicing point of view. In particular the slicing point of view leads to a further temporal derivative along the threading and along the test world lines. This slicing point of view is the foundation of *Black Holes: The Membrane Paradigm* by Thorne et al. [92] For black holes, the natural parametrized nonlinear reference frame is the one associated with Boyer-Lindquist coordinates, for which the zero-angular-momentum observers (ZAMO's) are the hypersurface normal observers, the static Killing observers are the threading observers, and the threading evolution vector field is the time coordinate derivative Killing vector field. For any parametrized nonlinear reference frame, one has $4 \times 3 = 12$ (hypersurface, threading, slicing contravariant, slicing covariant and "lie," "cfw," "fw") different decompositions of the 4-acceleration/4-force equation.

7 Spatial Conformal Transformations

For stationary spacetimes with a stationary family of observers and a stationary parametrized nonlinear reference frame (Lie dragged into itself by the symmetry group action) one may perform a conformal transformation of the spatial geometry on the observer quotient space which absorbs the spatial gravitational force term for massless test particles into the projected intrinsic derivative, leaving only the Coriolis or gravitomagnetic force remaining. (This can be done in any of the three points of view: threading, hypersurface, and slicing, although without stationarity, the part of the gravitoelectric force involving the Lie derivative of the shift remains present as well. This doubles the number of variations of the force decomposition to 24 for a parametrized nonlinear reference frame.) This gravitomagnetic force vanishes in a static spacetime with a static Killing observer congruence (for which all 12 variations of the acceleration decomposition coincide) so that the projection of null geodesics onto the observer quotient space leads to geodesics of the conformally rescaled spatial geometry, a fact known as Fermat's principle, well known for many decades.

Abramowicz et al have called this the "**optical geometry**" and Perlick the "**Fermat geometry**." This change in variables essentially shuffles terms in the measured acceleration/force equation between the 3-acceleration (longitudinal and transverse terms) and the spatial gravitoelectric force, apart from conformal rescalings.

In a stationary spacetime, the optical geometry is not itself directly linked to the paths of null geodesics, but in some sense to the average motion obtained by averaging the gravitomagnetic deflection in opposite directions of photon trajectories from the spatial geodesic paths. It is yet to be seen how this can help understand the physical properties of such spacetimes or of more general ones.

8 Difficulties from Not Using the Natural Mathematical Structure

8.1 Angular Velocity versus 3-Velocity in Stationary Axisymmetric Spacetimes

A minor inconvenience of the use of the angular velocity (connected with global behavior) to parametrize the description of local physical quantities is that one has no sense of the magnitude of the physical velocities involved, and graphs of various affects are distorted by the map between the (coordinate) angular velocity and angular physical component of the spatial velocity. By using instead the latter physical component with respect to natural observer families, comparisons are more clear, especially when one is interested in those velocities which correspond to timelike motion: the velocities which bound this interval have absolute value 1, while the corresponding bounding angular velocities are in general two complicated functions of the radius. However, since for example, De Felice or Semerák or Page do not directly refer to observer congruences but work with spacetime quantities, the angular velocity variable is natural for them. On the other hand, even in the spacetime discussion, the ZAMO's provide a natural way to convert angular velocities into physical spatial velocities in the angular direction in the entire exterior region of a black hole, and this new choice of variable can help better visualize the meaning of many graphs frequently displayed in their analyses.

In particular, the radial acceleration of the often studied equatorial plane circular orbits in a rotating black hole spacetime is a fractional quadratic function of the angular velocity Ω, or of de Felice's fractional linearly related y variable, or of the physical velocity in the angular direction ν with respect to any observers in circular motion, all variables which are related to each other by at most linear or fractional linear transformations. The simplest and most directly interpretable of all these variables seems to be the physical velocity in the angular direction with respect to the ZAMO observers, which are defined in the entire exterior of the hole. [83,44] Its use rectangularizes the region in the radius-velocity plane corresponding to timelike or lightlike motion (for which $\nu \in [-1, 1]$), removing the radial deformation of this region in the plane of the radius and an angular velocity related variable.

8.2 *Abramowicz et al and Spatial Projections*

The fundamental equations of determining the observer congruence in the Abramowicz approach are now [25]

$$g(n,n) = -1 \ , \quad \omega(n) = 0 \ , \quad a(n) = \nabla(n)\Phi \ . \tag{25}$$

Given any unit hypersurface-forming timelike vector field n on spacetime (the content of the first two equations), the third equation defines an acceleration potential for this observer congruence (discussed e.g., by Ehlers [91]) which is equivalent to a choice of lapse function N for the corresponding family of integral hypersurfaces of n, or equivalently a choice of time function t parametrizing this family, from which one may define the acceleration potential $\Phi = -\ln dt(n)$ (equivalent to $N = e^{\Phi}$).

Omitting the spatial projection of the gradient in the definition of the acceleration potential leads to the original version of the third Abramowicz fundamental equation

$$a(n) = \nabla\Phi \ . \tag{26}$$

Since $0 = n \cdot a(n) = \nabla_n \Phi$, it is clear that one must now impose the consistency condition $\pounds_n \Phi = 0$ (equivalently $\pounds_n N = 0$) on the acceleration potential and observer 4-velocity which limits the generality of the observer congruence. This older version only permits observer congruences for which the normal component $n \cdot X$ of any generating vector field X for a time function t parametrizing its family of integral hypersurfaces ($dt(X) = 1$) is Lie dragged along the congruence. In particular inertial observers of the Newtonian limit do not belong to the solution space of this older version of the fundamental equations as noted by Sonego and Massar, [24] while they are clearly permitted by the new fundamental equations which have an arbitrary timelike congruence as their solution.

There is also the question of the missing spatial projections in the decomposition of the spatial acceleration also noted by Sonego and Massar. In various discussions of the approach, the "**comoving frame of the test particle**" (namely LRS_U) has been brought into the discussion of the various terms in the decomposition of its 4-acceleration. However, in general only the centripetal acceleration term, however defined, lies in this local rest space and the observer local rest space. The longitudinal acceleration, however defined, lies in the observer local rest space, while the gravitational force, however defined, need not belong to either one unless spatially projected. The only consistent interpretation of the various force terms is to spatially project them into the local rest space of the observer. With the remarks of Sonego and Massar about the Newtonian limit, this spatial projection has finally been incorporated by Abramowicz et al. [25]

8.3 *Unnatural derivatives*

The use of an unnatural representation of derivatives along the test particle world line instead of one of the projected spatial intrinsic derivatives obscures all of the calculations of the Abramowicz et al approach and others like Semerák who have adopted the same notation when decomposing the 4-acceleration of a test particle world line. These derivatives had their origins in studying world lines which were

integral curves of Killing vector fields in stationary spacetimes, and so one could trivially extend vectors along the world lines to the entire relative observer world sheet and interpret derivatives along the direction of relative motion of these extended fields. This does not work for more general world lines in stationary spacetimes, and it requires detailed analysis to understand what preferred extensions simplify the resulting derivative.

Although formally ambiguous, one can almost guess how they were interpreted in their actual calculations. Later they realized this ambiguity when trying to extend their work to nonstationary spacetimes and tried to correct it by choosing a gauge condition to extend the relevant fields along the world line to the entire world sheet, but even this has had its problems complicated by the fact that sign problems plagued the inequivalent contravariant and covariant versions of this gauge condition which alternately appear in different discussions (index shifting does not commute with the Lie derivative).

Their intuition was clouded by the special features of transverse relative motion (circular orbits in stationary axisymmetric spacetimes), so the need for an additional spatial projection was not immediately understood. The same remains true of their current gauge condition, which is still not the natural one for a general spacetime.

They use a gauge condition to extend the spatial velocity off the test world line in order to define their optical spatial derivative

$$\tilde{\nabla}(n)_{\tilde{\nu}(U,n)}\,\tilde{\hat{\nu}}(U,n) \tag{27}$$

of the optical unit spatial velocity on the test world line in a direction not tangent to the world line on which it is defined. Interpreting how this gauge condition relates their formal derivatives to the various projected intrinsic derivatives is a formidable barrier to follow their calculations, especially since it seems to be a moving target in the Abramowicz et al force decomposition discussion.

The observer orthogonal decomposition of the Lie derivative along the rescaled observer 4-velocity field of a vector field X (orthogonally decomposed as $X = X^{\top}(n) + X(n)$) is

$$N^{-1}\pounds_{Nn}X = [\pounds_n X^{\top}(n)]n + \pounds(n)_n X(n)\ . \tag{28}$$

In order to give meaning to the spatial derivative $\nabla(n)$ of the spatial velocity unit vector $\hat{\nu}(U,n)$ of the test particle, the gauge condition of Lie dragging along Nn, namely $\pounds_{Nn}\hat{\nu}(U,n) = 0$, which implies

$$\pounds(n)_n\hat{\nu}(U,n) = \nabla_{(\mathrm{fw})}(n)\hat{\nu}(U,n) - \theta(n)\llcorner\hat{\nu}(U,n) = 0 \tag{29}$$

is used to extend to spatial velocity from the world line to the entire observer world sheet containing it. In fact this should only be done locally in some neighborhood of the world line since a world line which does not have unique intersections with each observer world line it meets cannot in general be compatible with such a global extension.

However, the more natural gauge condition on the local extension of the (contravariant) optical unit spatial velocity

$$\begin{aligned}&\nabla_{(\mathrm{lie})}(n)\tilde{\hat{\nu}}(U,n) = \pounds(n)_n\tilde{\hat{\nu}}(U,n) = 0 \Rightarrow\\ &\tilde{D}_{(\mathrm{lie})}(U,n)\,\tilde{\hat{\nu}}(U,n)^{\beta}/d\tilde{\ell}_{(U,n)} = \tilde{\nabla}(u)_{\tilde{\hat{\nu}}(U,n)}\,\tilde{\hat{\nu}}(U,n)\end{aligned} \tag{30}$$

makes their spatial derivative of this quantity agree with the contravariant version of the natural optical spatial Lie intrinsic derivative with respect to the observer optical spatial arc length parametrization (see (A.20) of Bini et al [43])

$$
\begin{aligned}
&\tilde{D}_{(\text{lie})}(U,n)\,\tilde{\hat{\nu}}(U,n)^\beta/d\tilde{\ell}_{(U,n)} \\
&\quad = [1/\tilde{\nu}(U,n)][\nabla_{(\text{lie})}(n) + \tilde{\nu}(U,n)\tilde{\hat{\nu}}(U,n)^\alpha\tilde{\nabla}(n)_\alpha]\tilde{\hat{\nu}}(U,n)^\beta \\
&\quad = N^2[D_{(\text{lie})}(U,n)\,\hat{\nu}(U,n)^\beta/d\ell_{(U,n)} \\
&\quad\ +P_n(U,n)^{(\perp)\beta}{}_\alpha\nabla(n)^\alpha\ln N + \hat{\nu}(U,n)^\beta/\nu(U,n)\,\nabla_{(\text{lie})}(n)\ln N]\ , \qquad (31)
\end{aligned}
$$

where the first equality defines the equivalent action of the new derivative on a congruence of test particle world lines, while the second defines it for a single such world line, and $\tilde{\nabla}(n)$ is the optical spatial covariant derivative satisfying $\tilde{\nabla}(n)\tilde{h}(n) = 0$. Their current condition leads to additional terms which only vanish for motion along Killing directions in a stationary spacetime referred to a stationary observer congruence. With our present choice of gauge condition, their final decomposition of forces becomes the hypersurface point of view optical decomposition (15.7) of Bini et al, [43] with their Coriolis force term coinciding with the gravitomagnetic term which in this case is due to the expansion of the observer congruence alone.

Following their steps (and continuing revisions) requires some patience, but the end result is clear: their goal is just one of the 12 optical possibilities outlined above in the more general setting (including observer 4-velocity fields with nonzero vorticity and the bi-congruence slicing point of view) that follow cleanly from performing a space-plus-time split of spacetime, accompanied by the natural Fermat's principle spatial conformal transformation.

9 Cumulative Drag Index

Prasanna and Iyer [28,29] have introduced a "**cumulative drag index**" for circular orbits in the equatorial plane of the Kerr spacetime in an effort to find some intrinsic characterization of spacetimes with rotation. In their notation for the optical decomposition of the radial acceleration of such a circular orbit, this index is

$$
\mathcal{C} = \frac{Cf + Co - Gr}{Cf + Co + Gr} \qquad (32)
$$

or

$$
(1-\mathcal{C})/2 = \frac{Gr}{Cf + Co + Gr}\ , \qquad (33)
$$

where Co, Cf, and Gr are the centrifugal force, optical Coriolis force, and optical gravitational force in the hypersurface point of view, modulo sign (the acceleration term versus inertial force sign). This may be rewritten in the gravitoelectromagnetism notation in terms of orthonormal components with respect to the orthogonal Boyer-Lindquist spatial coordinate frame (using the associated parametrized nonlinear reference frame) as

$$
(1-\mathcal{C})/2 \quad = \quad \frac{g(n)^{\hat{r}}}{a(U)^{\hat{r}}} = \frac{\mathcal{F}(0;\kappa,\nu_+,\nu_-)}{\mathcal{F}(\nu;\kappa,\nu_+,\nu_-)}
$$

$$= \frac{\nu_+\nu_-(1-\nu^2)}{(\nu-\nu_-)(\nu-\nu_+)} \ , \tag{34}$$

where $\kappa = \kappa(U,n)^{\hat\phi}$, $\nu_\pm = \nu(U_\pm,n)^{\hat\phi}$, $\nu = \nu(U,n)^{\hat\phi}$, and $U_\pm$ are the geodesic 4-velocities and where the physical component of the radial acceleration is

$$\begin{aligned} a(U)^{\hat r} &= \mathcal{F}(\nu;\kappa,\nu_+,\nu_-) = \kappa\frac{(\nu-\nu_-)(\nu-\nu_+)}{(1-\nu^2)} \\ &= \gamma^2[\kappa\nu^2 - g(n)^{\hat r} - H_{(\mathrm{fw})}(n)^{\hat r}{}_{\hat\phi}\nu] \ . \end{aligned} \tag{35}$$

The zero's of the denominator of the quantity $(1-\mathcal{C})/2$ coincide with those of $\mathcal{C}$ and occur at the geodesic velocities. The zero's of the numerator of this quantity are qualitatively similar to the zeros of $\mathcal{C}$ but in contrast have geometric meaning as the light velocities. The gamma factor $\gamma^{-2} = 1-\nu^2$ leading to those zeros may be removed from this quantity by instead taking the ratio of the part of the total acceleration due to the gravitoelectric force and the total acceleration itself

$$\gamma^2(1-\mathcal{C})/2 = \frac{-\gamma^2 g(n)^{\hat r}}{a(U)^{\hat r}} = \frac{\nu_-\nu_+}{(\nu-\nu_-)(\nu-\nu_+)} \ . \tag{36}$$

One factor of gamma in this formula is part of the spatial gravitational force, while the second factor corresponds to the conversion back to the test particle proper time force component, in order to be compared to the test particle 3-acceleration. In a similar way one could also introduce the ratios of the gravitomagnetic or centripetal force terms to the total radial acceleration, yielding a total of three new indices measuring the relative importance of these three contributions to the total acceleration. However, but they are all infinite for geodesics and do not seem to have a particularly intrinsic meaning apart from the fact that the ZAMO observers are geometrically defined by the symmetry of the spacetime.

10 Towards a Better Understanding of the Arena of Classical General Relativity

We have no interest in attacking personalities or their work. The fact that after all these years, such confusion can still prevail in a topic like this shows the importance of clarifying the basic tools we are using. All the formalism of gravitoelectromagnetism, the "relativity of spacetime splittings," is not our invention, nor that of all the others who have worked on various aspects of it over the past half century. (References are given elsewhere. [32]) It flows naturally from the mathematical foundations of general relativity. One only has to respect the natural mathematical structure already present, and use a notation which unambiguously describes all the possible variations that are allowed by that structure. The utility of any of the approaches which fall under this umbrella depends on the particular application and what advantages in understanding it can convey to us. Certainly introducing complication in an existing spacetime which does not help us appreciate its properties better than a direct spacetime approach is not useful, but in many situations, a space-plus-time perspective does yield useful information.

We appreciate all of the work done on this topic in static spacetimes, where the elegance of the application clearly provides the motivation to extend it to stationary and more general spacetimes. Indeed one may extend the optical geometry approach in a consistent way not only to arbitrary hypersurface-forming timelike vector fields in any spacetime, but to any arbitrary timelike vector field, or to the context of the slicing approach of ADM. However, unless we are able to communicate with each other in a clear way, any advantages of one approach or another will remain locked in that approach without the possibility of others appreciating its meaning or translating it into their language. Our ultimate goal after all is to shed more light on a rather complicated theory, in a way that we can all benefit. We hope that the ideas described in the present article may contribute to a better understanding among us about what we are all doing and what it all means in relationship to each of our perspectives.

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